J. Math. Anal. Appl. 348 (2008) 760–774

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Compact trace in weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega; v_0, v_1)$ ☆

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ARTICLE INFO

Article history:
Received 21 January 2008
Available online 7 August 2008
Submitted by A. Cianchi

Keywords:
Weighted variable exponent Lebesgue space
Weighted variable exponent Sobolev space
Compact embedding
Compact trace
Extension operator

ABSTRACT

We study trace operators in weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega; v_0, v_1) \hookrightarrow L^{q(x)}(\partial\Omega; w)$ for sufficiently regular unbounded domain $\Omega \subseteq \mathbb{R}^N$ ($N \geqslant 2$) with noncompact boundary, where $p(x)$ is a Lipschitz continuous function defined on Ω satisfying $1 < p^- \leqslant p^+ < N$. We show that when $\operatorname{ess\,inf}_{x\in\Omega} (\frac{N-1}{q(x)} - \frac{N}{p(x)} + 1) > 0$, the trace operators $W^{1,p(x)}(\Omega; v_0, v_1) \hookrightarrow L^{q(x)}(\partial\Omega; w)$ are compact under certain conditions on weight functions v_0, v_1, w.

Published by Elsevier Inc.

1. Introduction

In the study of partial differential equations and variational problems with $p(x)$-growth conditions in a bounded domain $\Omega \subseteq \mathbb{R}^N$ it is often useful to know that the embeddings of the Sobolev space $W^{m,p(x)}(\Omega)$ into the Lebesgue space $L^{q(x)}(\Omega)$ and the trace operators of the $W^{m,p(x)}(\Omega)$ into $L^{q(x)}(\partial\Omega)$ are compact (see [2,10,11,13–16,18,19]). In particular for variational problem such a compactness property is often used to show that the energy functional for this problem satisfies the Palais–Smale condition (see [17,26–28]). But when Ω is unbounded, compactness of the embeddings and traces above fails in general. So it seems to be natural to study more general function spaces, for instance, weighted Sobolev spaces, where compact embeddings and traces can be obtained for suitable weight functions (see [20,29]), when $p(x) \equiv p$, there are many study papers (see [1,3,21–25]). The aim of the present paper is to extend the main results of [21] to variable exponent case, under certain conditions on weight functions, we obtain the compactness of embeddings and traces in $W^{1,p(x)}(\Omega; v_0, v_1)$, which is very useful for study the variational problems with $p(x)$-growth on unbounded domain Ω, especially for nonlinear boundary value problem with $p(x)$-growth on unbounded domain Ω, while $p(x) \equiv p$ (see [5,24,25]).

This paper is divided into four sections. In Section 2, we first derive certain necessary and sufficient conditions for the compactness of trace operator $X(\mathbb{R}^N) \hookrightarrow Y(\Gamma)$ between two Banach function spaces $X(\mathbb{R}^N)$ and $Y(\Gamma)$, where Γ is a smooth $(N-k)$-dimensional submanifold of $\mathbb{R}^N$ (Theorem 2.1), which follows the ideas of M. Krbec, B. Opic and L. Pick's paper [20] and the B. Opic and A. Kufner's book [21], it is the basic throughout the present paper, then, we recall some basic facts about the weighted variable exponent Lebesgue and Sobolev spaces. In Section 3, we give the main results: trace theorem (Theorem 3.1). In order to prove our main results, we first study the extension theorem of $W^{1,p(x)}(\Omega; v_0, v_1)$ (Theorem 3.2), so, we can transfer our trace theorem in $\mathbb{R}^N$ to more general domain Ω, then we prove a special case of our main result (Theorem 3.3) and obtain a corollary (Corollary 3.1), at last, we give the proof of our main results. In Section 4 we give an example of our main results.

☆ Research supported by the National Natural Science Foundation of China (10671084).
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0022-247X/$ – see front matter Published by Elsevier Inc.
doi:10.1016/j.jmaa.2008.08.004

2. Preliminaries

2.1. Notations about trace

Let $D_n \subseteq \mathbb{R}^N$ be an increasing sequence of bounded domains in $\mathbb{R}^N$, such that $\mathbb{R}^N = \bigcup_n D_n$, and denote $D^n = \mathbb{R}^N \backslash D_n$.

Let Γ be a smooth $(N-k)$-dimensional submanifold of $\mathbb{R}^N$, and denote $\Gamma_n = D_n \cap \Gamma$, $\Gamma^n = \Gamma \backslash \Gamma_n$.

Let $X(\mathbb{R}^N)$ and $Y(\Gamma)$ be two Banach spaces of functions defined on $\mathbb{R}^N$ and Γ, and assume throughout that Y has an absolutely continuous and monotone norm (by absolutely continuous we mean that for every nonincreasing sequence of measurable sets $G_n \subseteq \Gamma$ with $|G_n| \to 0$ we have $\|u\chi_{G_n}\|_{Y,\Gamma} \to 0$, where $|G_n|$ denote the $(N-k)$-dimensional Lebesgue measure of G_n, and by the monotonicity we mean that the relation $0 \leqslant u_1 \leqslant u_2$ a.e. in Γ implies $\|u_1\|_{Y,\Gamma} \leqslant \|u_2\|_{Y,\Gamma}$). Denote by $X(D_n)$ the sets of restriction to D_n of functions from $X(\mathbb{R}^N)$, and denote $X(D^n)$, $Y(\Gamma_n)$ and $Y(\Gamma^n)$ in a similar way. Respectively, the space $Y(\Gamma_n)$ is equipped with the norm $\|u\|_{Y,\Gamma_n} = \|u\chi_{\Gamma_n}\|_{Y,\Gamma}$, where χ_{Γ_n} is the characteristic function of the set Γ_n.

The space $X(D_n)$ is supposed to be endowed with a norm $\|\cdot\|_{X,D_n}$ only satisfying

$$\|u\|_{X,D_n} \leqslant C\|u\|_{X,\mathbb{R}^N} \tag{2.1}$$

with an appropriate constant C independent of u.

As usual, let $\gamma : X(\mathbb{R}^N) \to Y(\Gamma)$ be the trace operator, and we denote a function u defined in $\mathbb{R}^N$ and its trace $\gamma(u)$ on Γ by the same symbol u, other properties of trace operator can be seen in [8]. Our main tool in the proof of our main results below is the following.

Theorem 2.1. *Assume*

$$X(D_n) \hookrightarrow Y(\Gamma_n) \quad \text{is compact for every } n \in \mathbb{N}, \tag{2.2}$$

$$\lim_{n\to\infty} \sup_{\|u\|_{X,\mathbb{R}^N} \leqslant 1} \|u\|_{Y,\Gamma^n} = 0, \tag{2.3}$$

then the trace operator $X(\mathbb{R}^N) \hookrightarrow Y(\Gamma)$ is compact. On the other hand, if the trace operator $X(\mathbb{R}^N) \hookrightarrow Y(\Gamma)$ is compact, then (2.3) *holds.*

If

$$X(D_n) \hookrightarrow Y(\Gamma_n) \quad \text{is continuous for every } n \in \mathbb{N}, \tag{2.4}$$

$$\lim_{n\to\infty} \sup_{\|u\|_{X,\mathbb{R}^N} \leqslant 1} \|u\|_{Y,\Gamma^n} < \infty, \tag{2.5}$$

then the trace operator $X(\mathbb{R}^N) \hookrightarrow Y(\Gamma)$ is continuous. If the trace operator $X(\mathbb{R}^N) \hookrightarrow Y(\Gamma)$ is continuous, then (2.5) *holds.*

Proof. First of all, we notice that the statement (2.3) is equivalent to the statement that for every $\varepsilon > 0$, there exists $\tilde{n} \in \mathbb{N}$, such that for all $u \in X(\mathbb{R}^N)$,

$$\|u\|_{Y,\Gamma} \leqslant \varepsilon\|u\|_{X,\mathbb{R}^N} + \|u\|_{Y,\Gamma_{\tilde{n}}}. \tag{2.6}$$

Indeed, let $\varepsilon > 0$ and (2.3) hold, there is some $\tilde{n} \in \mathbb{N}$ such that for every $n \geqslant \tilde{n}$, we have

$$\|u\|_{Y,\Gamma^n} \leqslant \varepsilon\|u\|_{X,\mathbb{R}^N}$$

and in turn

$$\|u\|_{Y,\Gamma^n} = \|u\chi_{\Gamma^n}\|_{Y,\Gamma} = \big\|u(1-\chi_{\Gamma_n})\big\|_{Y,\Gamma} \geqslant \|u\|_{Y,\Gamma} - \|u\|_{Y,\Gamma_n},$$

then, we obtain (2.6). On the other hand, if (2.6) is satisfied for every $\varepsilon > 0$, we can choose $\tilde{n}$, such that

$$\|u\|_{Y,\Gamma^{\tilde{n}}} + \|u\|_{Y,\Gamma_{\tilde{n}}} \leqslant \varepsilon\|u\|_{X,\mathbb{R}^N} + \|u\|_{Y,\Gamma_{\tilde{n}}},$$

which implies

$$\|u\|_{Y,\Gamma^n} \leqslant \varepsilon\|u\|_{X,\mathbb{R}^N} \quad \text{for every } n \geqslant \tilde{n}.$$

Now, let $\{u_j\}$ be bounded in $X(\mathbb{R}^N)$, we shall distinguish two case. First, if there is an infinite subsequence $\{u_{j_k}\} \subseteq \{u_j\}$ such that $\|u_{j_k}\|_{X,\mathbb{R}^N} = 0$, then according to (2.1), we have $\|u_{j_k}\|_{X,D_n} = 0$ for all $n \in \mathbb{N}$ and in virtue of (2.2), we have $\|u_{j_k}\|_{Y,\Gamma_n} = 0$. As $\Gamma = \bigcup_n \Gamma_n$, clearly $\|u_{j_k}\|_{Y,\Gamma} = 0$ and thus we have found a convergent subsequence of $\{u_j\}$ in $Y(\Gamma)$. In the second case, we can assume that $\|u_j\|_{X,\mathbb{R}^N} > 0$ for all $j \in \mathbb{N}$, let $\varepsilon > 0$ and $\tilde{n}$ be the corresponding number from (2.6), the condition (2.1) implies the boundedness of $\{\|u_j\|_{X,D_{\tilde{n}}}\}_{j=1}^{\infty}$, and by (2.2), we can suppose $u_j \to u$ in $Y(\Gamma_{\tilde{n}})$ with no loss of generality, and we have

$$\|u_j - u_i\|_{Y,\Gamma} \leqslant \varepsilon\|u_j - u_i\|_{X,\mathbb{R}^N} + \|u_j - u_i\|_{Y,\Gamma_{\bar{n}}} \leqslant (2C+1)\varepsilon,$$

where C is the bound for $\|u_j\|_{X,\mathbb{R}^N}$, and i, j are sufficiently large, so $\{u_j\}$ contains a Cauchy sequence in $Y(\Gamma)$ and the trace operator is compact.

On the other hand, let $X(\mathbb{R}^N) \hookrightarrow Y(\Gamma)$ be compact and assume that (2.6) does not hold, that is

$$\|u_n\|_{Y,\Gamma} > \varepsilon_0\|u_n\|_{X,\mathbb{R}^N} + \|u\|_{Y,\Gamma_n},$$

for some sequence $\{u_n\}$, $\|u_n\|_{X,\mathbb{R}^N} \neq 0$ and some $\varepsilon_0 > 0$. Put $v_n = \frac{u_n}{\|u_n\|_{X,\mathbb{R}^N}}$, then we obtain

$$\|v_n\|_{Y,\Gamma} \geqslant \varepsilon_0 + \|v_n\|_{Y,\Gamma_n},$$

take the limit in the above inequality, that is

$$\|v\|_{Y,\Gamma} \geqslant \varepsilon_0 + \|v\|_{Y,\Gamma},$$

which is a contradiction.

So, the proof of the first part of this theorem is complete, and the second part can be proved in a similar way. □

Remark 2.1. Under the assumption (2.2), the condition (2.3) is necessary and sufficient for the compact trace $X(\mathbb{R}^N) \hookrightarrow Y(\Gamma)$; under the assumption (2.4), the condition (2.5) is necessary and sufficient for the continuous trace $X(\mathbb{R}^N) \hookrightarrow Y(\Gamma)$.

2.2. Notations about $W^{1,p(x)}(\Omega; v_0, v_1)$

Let $\Omega \subseteq \mathbb{R}^N$ be a domain with nonempty boundary $\partial\Omega$, denote

$$L_+^\infty(\Omega) = \Big\{p \in L^\infty(\Omega)\colon \operatorname*{ess\,inf}_{x\in\Omega} p(x) > 1\Big\}.$$

For $p \in L_+^\infty(\Omega)$, denote

$$p^- = p^-(\Omega) = \operatorname*{ess\,inf}_{x\in\Omega} p(x), \qquad p^+ = p^+(\Omega) = \operatorname*{ess\,inf}_{x\in\Omega} p(x).$$

Let w, v_0, v_1 are measurable nonnegative and a.e. finite functions defined in $\mathbb{R}^N$. For $p \in L_+^\infty(\Omega)$, define

$$L^{p(x)}(\Omega; w) = \left\{u\colon\ u(x) \text{ is a measurable function on } \Omega \text{ and } \int_\Omega w(x)\big|u(x)\big|^{p(x)}\,dx < \infty\right\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega;w)} = |u|_{p(x),\Omega,w} = \inf\left\{\lambda > 0: \int_\Omega w(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\}.$$

When $w(x) \equiv 1$, we use $L^{p(x)}(\Omega)$ instead of $L^{p(x)}(\Omega; w)$ and use $|u|_{p(x),\Omega}$ instead of $|u|_{p(x),\Omega,w}$.

Define

$$W^{1,p(x)}(\Omega; v_0, v_1) = \big\{u \in L^{p(x)}(\Omega; v_0)\colon\ \big|\nabla u(x)\big| \in L^{p(x)}(\Omega; v_1)\big\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega;v_0,v_1)} = \|u\|_{1,p(x),\Omega,v_0,v_1} = |u|_{p(x),\Omega,v_0} + |\nabla u|_{p(x),\Omega,v_1}.$$

When $v_0(x) \equiv v_1(x) \equiv 1$, we use $W^{1,p(x)}(\Omega)$ instead of $W^{1,p(x)}(\Omega; v_0, v_1)$ and use $\|u\|_{1,p(x),\Omega}$ instead of $\|u\|_{1,p(x),\Omega,v_0,v_1}$. Also, it is easy to see that the norm

$$|||u|||_{1,p(x),\Omega,v_0,v_1} = \inf\left\{\lambda > 0\colon \int_\Omega \left(v_0(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} + v_1(x)\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)}\right) dx \leqslant 1\right\}$$

is the equivalent norm.

We assume throughout this paper that our weight functions w, v_0, v_1 belong to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$; and we also assume that for all cubes Q in $\mathbb{R}^N$, we have $\max_{x\in Q}\{v\} \leqslant C(\ell(Q)) \min_{x\in Q}\{v\}$, where $\ell(Q)$ denotes the edge length of the cube Q, $C(\ell(Q))$ is a positive constant depends on $\ell(Q)$. By analysis, we can easy find that the function $v(x) = \frac{1}{(1+|x|)^{\alpha(x)}}$ (where $\alpha(x)$ is a continuous function defined in $\mathbb{R}^N$ and satisfies $-N < \alpha(x) < N(p(x)-1)$, $|x|$ denotes the maximum norm in $\mathbb{R}^N$) can be seen as a weight functions; for other properties of weight functions, see [30].

As usual, we also denote $p_1(x) \ll p_2(x)$ in Ω the fact that

$$\operatorname*{ess\,inf}_{x\in\Omega}\big(p_2(x) - p_1(x)\big) > 0.$$

On the basic properties of the spaces $L^{p(x)}(\Omega; w)$ and $W^{1,p(x)}(\Omega; v_0, v_1)$, we refer to [9,10,13,15,17]. Here we display some facts which will be used later.

Proposition 2.1. *(See [15,17,19].) The spaces $L^{p(x)}(\Omega; w)$ and $W^{1,p(x)}(\Omega; v_0, v_1)$ are separable and reflexive Banach spaces.*

Proposition 2.2. *(See [15,17,19].) Set $\phi(u) = \int_\Omega w(x)|u(x)|^{p(x)}\,dx$, for $u, u_k \in L^{p(x)}(\Omega; w)$, we have*

(1) *for* $u \neq 0$, $|u|_{p(x),\Omega,w} = \lambda \Leftrightarrow \phi(\frac{u}{\lambda}) = 1$;
(2) $|u|_{p(x),\Omega,w} < 1$ $(=1;\ >1) \Leftrightarrow \phi(u) < 1$ $(=1;\ >1)$;
(3) *if* $|u|_{p(x),\Omega,w} > 1$, *then* $|u|^{p^-}_{p(x),\Omega,w} \leqslant \phi(u) \leqslant |u|^{p^+}_{p(x),\Omega,w}$;
(4) *if* $|u|_{p(x),\Omega,w} < 1$, *then* $|u|^{p^+}_{p(x),\Omega,w} \leqslant \phi(u) \leqslant |u|^{p^-}_{p(x),\Omega,w}$;
(5) $\lim_{k\to\infty} |u_k|_{p(x),\Omega,w} = 0 \Leftrightarrow \lim_{k\to\infty} \phi(u_k) = 0$;
(6) $|u_k|_{p(x),\Omega,w} \to \infty \Leftrightarrow \phi(u_k) \to \infty$.

Similar to Proposition 2.2, we have

Proposition 2.3. *Set $I(u) = \int_\Omega (v_0(x)|u(x)|^{p(x)} + v_1(x)|\nabla u(x)|^{p(x)})\,dx$, for $u, u_k \in W^{1,p(x)}(\Omega; v_0, v_1)$, then*

(1) *for* $u \neq 0$, $|||u|||_{1,p(x),\Omega,v_0,v_1} = \lambda \Leftrightarrow I(\frac{u}{\lambda}) = 1$;
(2) $|||u|||_{1,p(x),\Omega,v_0,v_1} < 1$ $(=1;\ >1) \Leftrightarrow I(u) < 1$ $(=1;\ >1)$;
(3) *if* $|||u|||_{1,p(x),\Omega,v_0,v_1} > 1$, *then* $|||u|||^{p^-}_{1,p(x),\Omega,v_0,v_1} \leqslant I(u) \leqslant |||u|||^{p^+}_{1,p(x),\Omega,v_0,v_1}$;
(4) *if* $|||u|||_{1,p(x),\Omega,v_0,v_1} < 1$, *then* $|||u|||^{p^+}_{1,p(x),\Omega,v_0,v_1} \leqslant I(u) \leqslant |||u|||^{p^-}_{1,p(x),\Omega,v_0,v_1}$;
(5) $\lim_{k\to\infty} |||u_k|||_{1,p(x),\Omega,v_0,v_1} = 0 \Leftrightarrow \lim_{k\to\infty} I(u_k) = 0$;
(6) $|||u_k|||_{1,p(x),\Omega,v_0,v_1} \to \infty \Leftrightarrow I(u_k) \to \infty$.

Proposition 2.4. *If $1 < p(x) \leqslant q(x) < +\infty$, $0 < w(x) \leqslant v(x)$ for a.e. $x \in \Omega$, and $|\Omega| < \infty$, then*

$$|u|_{p(x),\Omega,w} \leqslant C|u|_{q(x),\Omega,v},$$

where C is independent of u.

Proof. Set $\lambda = |u|_{q(x),\Omega,v}$ and $\Omega_0 = \{x \in \Omega\colon\ |u(x)| \leqslant \lambda\}$, for $w, v \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, then there is $C_1 > 0$ such that $\int_{\Omega_0} v(x)\,dx < C_1$, from Proposition 2.2, setting $C > 1$, we have

$$\begin{aligned}
\int_\Omega w(x)\left|\frac{u(x)}{C\lambda}\right|^{p(x)} dx &\leqslant \int_{\Omega_0} w(x)\left|\frac{u(x)}{C\lambda}\right|^{p(x)} dx + \int_{\Omega\setminus\Omega_0} w(x)\left|\frac{u(x)}{C\lambda}\right|^{p(x)} dx \\
&\leqslant \int_{\Omega_0} v(x)\left|\frac{u(x)}{C\lambda}\right|^{p(x)} dx + \int_{\Omega\setminus\Omega_0} v(x)\left|\frac{u(x)}{C\lambda}\right|^{p(x)} dx \\
&\leqslant \frac{1}{C^{p^-}}\int_{\Omega_0} v(x)\,dx + \frac{1}{C^{p^-}}\int_\Omega v(x)\left|\frac{u(x)}{\lambda}\right|^{q(x)} dx \\
&\leqslant \frac{C_1}{C^{p^-}} + \frac{1}{C^{p^-}} = \frac{C_1+1}{C^{p^-}},
\end{aligned}$$

when C is big enough, we have $\frac{C_1+1}{C^{p^-}} \leqslant 1$, and the proof is complete. $\square$

Proposition 2.5. *Let $x_0 \in \mathbb{R}^N$ and $\rho > 0$, set $B(x_0,\rho) = \{y \in \mathbb{R}^N\colon\ |x_0 - y| < \rho\}$. Let $u \in L^{p(x)}(B(x_0,\rho))$ and put $\tilde{u}(y) = u(x_0 + \rho y)$, $\tilde{p}(y) = p(x_0 + \rho y)$ for $y \in B(0,1)$, then there exists a constant $C \geqslant 1$, such that for any $x \in B(x_0,\rho)$ we have*

$$C^{-1}\rho^{\frac{N}{p(x)}}|\tilde{u}|_{\tilde{p}(y),B(0,1)} \leqslant |u|_{p(x),B(x_0,\rho)} \leqslant C\rho^{\frac{N}{p(x)}}|\tilde{u}|_{\tilde{p}(y),B(0,1)},$$
$$C^{-1}\rho^{\frac{N}{p(x)}-1}|\nabla\tilde{u}|_{\tilde{p}(y),B(0,1)} \leqslant |\nabla u|_{p(x),B(x_0,\rho)} \leqslant C\rho^{\frac{N}{p(x)}-1}|\nabla\tilde{u}|_{\tilde{p}(y),B(0,1)}.$$

Proof. From the Proposition 2.2, we find

$$\rho_-|\tilde{u}|_{\tilde{p}(y),B(0,1)} \leqslant |u|_{p(x),B(x_0,\rho)} \leqslant \rho_+|\tilde{u}|_{\tilde{p}(y),B(0,1)},$$

where $\rho_+ = \operatorname{ess\,sup}_{x\in B(x_0,\rho)}\{\rho^{\frac{N}{p(x)}}\} = \max\{\rho^{\frac{N}{p^+}}, \rho^{\frac{N}{p^-}}\}$, $\rho_- = \operatorname{ess\,sup}_{x\in B(x_0,\rho)}\{\rho^{\frac{N}{p(x)}}\} = \min\{\rho^{\frac{N}{p^+}}, \rho^{\frac{N}{p^-}}\}$. Set $C \geqslant \frac{\rho_+}{\rho_-} \geqslant 1$, then there is

$$C^{-1}\rho^{\frac{N}{p(x)}} \leqslant \rho_- \leqslant \rho_+ \leqslant C\rho^{\frac{N}{p(x)}}.$$

So we have

$$C^{-1}\rho^{\frac{N}{p(x)}}|\tilde{u}|_{\tilde{p}(y),B(0,1)} \leqslant |u|_{p(x),B(x_0,\rho)} \leqslant C\rho^{\frac{N}{p(x)}}|\tilde{u}|_{\tilde{p}(y),B(0,1)}$$

and the proof of the first inequalities is complete.

The second inequalities can be proved in a similar way. □

Proposition 2.6. *Let U, V are two bounded domains in $\mathbb{R}^N$ and $\varphi : U \to V$ be a C^1-diffeomorphism and the partial derivative of the coordinate functions φ and φ^{-1} are uniformly bounded by a positive constant K. For $x \in U$, let $u \in L^{p(x)}(U)$, $y = \varphi(x)$, and put $\tilde{u}(y) = u(\varphi^{-1}(y))$, $\tilde{p}(y) = p(\varphi^{-1}(y))$ for $y \in V$, then there exists a constant $C > 0$ which is independent of u, such that*

$$C^{-1}|\tilde{u}|_{\tilde{p}(y),V} \leqslant |u|_{p(x),U} \leqslant C|\tilde{u}|_{\tilde{p}(y),V},$$
$$C^{-1}|\nabla\tilde{u}|_{\tilde{p}(y),V} \leqslant |\nabla u|_{p(x),U} \leqslant C|\nabla\tilde{u}|_{\tilde{p}(y),V}.$$

Proof. Since there hold

$$\int_U |u(x)|^{p(x)}\,dx = \int_V |\tilde{u}(y)|^{\tilde{p}(y)}\,d\varphi^{-1}(y) \leqslant K^N \int_V |\tilde{u}(y)|^{\tilde{p}(y)}\,dy,$$

and

$$\int_V |\tilde{u}(y)|^{\tilde{p}(y)}\,dy = \int_U |u(x)|^{p(x)}\,d\varphi(x) \leqslant K^N \int_U |u(x)|^{p(x)}\,dx,$$

we can easily get the proof of the first inequalities, the second one can be proved in a similar way. □

Proposition 2.7. *Let $u \in L^{p(x)}(B(x_0,\rho); w)$, and suppose that there is a real constant $w_0 > 0$, such that $|w(x)|^{\frac{1}{p(x)}} \geqslant w_0 > 0$ for all $x \in B(x_0,\rho)$, then we have*

$$w_0|u|_{p(x),B(x_0,\rho)} \leqslant |u|_{p(x),B(x_0,\rho),w}.$$

Proof. Set $\lambda = |u|_{p(x),B(x_0,\rho),w}$, then

$$\int_{B(x_0,\rho)} \left|\frac{w_0 u(x)}{\lambda}\right|^{p(x)} dx \leqslant \int_{B(x_0,\rho)} w(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx = 1,$$

that is $w_0|u|_{p(x),B(x_0,\rho)} \leqslant \lambda$, and the proof is complete. □

Proposition 2.8. *(See [10,13,15,18].) Let Ω be a bounded domain in $\mathbb{R}^N$, $p \in C^{0,1}(\overline{\Omega})$, $1 < p^- \leqslant p^+ < N$. Then:*

for any $q \in L_+^\infty(\Omega)$ with $q(x) \leqslant \frac{Np(x)}{N-p(x)}$, there is continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, and when $q(x) \ll \frac{Np(x)}{N-p(x)}$, the embedding is compact;
for any $q \in C(\partial\Omega)$ with $1 \leqslant q(x) \leqslant \frac{(N-1)p(x)}{N-p(x)}$, there is a continuous trace $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$, when $1 \leqslant q(x) \ll \frac{(N-1)p(x)}{N-p(x)}$, the trace is compact.

3. Main results and proofs

We define a covering of $\mathbb{R}^N$ by family of cubes with edge length $2n$,

$$D_n = \{x \in \mathbb{R}^N\colon\ |x| < n\},$$

and define

$$D^n = \mathbb{R}^N \setminus D_n$$

($|\cdot|$ denotes the maximum norm in $\mathbb{R}^N$). For $\Omega \subseteq \mathbb{R}^N$, $\Gamma = \partial\Omega$, denote $\Omega_n = D_n \cap \Omega$, $\Gamma_n = D_n \cap \Gamma$. In this paper, we always suppose that Ω is an unbounded domain with sufficiently smooth boundary $\Gamma = \partial\Omega$ (the exact assumptions on Γ are formulated later).

For weights, we assume further that there exist a bounded positive continuous function r and positive continuous functions b_0, b_1 defined in $\mathbb{R}^N$, and constants K_1, C_r, such that for some $\tilde{n} \geqslant 2$, the following conditions hold:

$$r(x) \leqslant \big(|x|+1\big)/3 \qquad \text{for every } x \in D^{\tilde{n}}; \tag{3.1}$$

$$C_r^{-1} \leqslant r(x)/r(z) \leqslant C_r \qquad \text{for every } x \in D^{\tilde{n}},\ z \in Q\big(x, r(x)\big); \tag{3.2}$$

$$\big|v_1(x)\big|^{\frac{1}{p(x)}} r^{-1}(x) \leqslant K_1 \big|v_0(x)\big|^{\frac{1}{p(x)}} \qquad \text{for a.e. } x \in D^{\tilde{n}}; \tag{3.3}$$

$$\big|w(z)\big|^{\frac{1}{p(z)}} \leqslant b_0(x), b_1(x) \leqslant \big|v_1(z)\big|^{\frac{1}{p(z)}} \qquad \text{for every } x \in D^{\tilde{n}},\ \text{a.e. } z \in Q\big(x, r(x)\big), \tag{3.4}$$

where $Q(x, r(x))$ denotes the cube in $\mathbb{R}^N$ which with the center at x and with edge length $2r(x)$, by K. Pflüger's [22,24] or analysis, we know $Q(x, r(x)) \cap D^{3n} \neq \emptyset$ and $n \geqslant \tilde{n}$ imply $Q(x, r(x)) \subseteq D^n$. We also denote $Q^{\gamma}(x, r(x)) = Q(x, r(x)) \cap \Gamma$.

For the boundary $\Gamma = \partial\Omega$, we assume that there exists a locally finite covering of Γ with open subsets $U_i \subset \mathbb{R}^N$ having the following properties:

(U1) There is a global constant θ such that $\sum_i \chi_{U_i}(z) \leqslant \theta$ for every $z \in \mathbb{R}^N$;
(U2) There exist cubes $B_i \subseteq \mathbb{R}^N$ and Lipschitz-diffeomorphisms $\varphi_i : U_i \to B_i$ such that $0 \in B_i$ and $\varphi_i^{-1}(\mathbb{R}^{N-1} \times \{0\}) = U_i \cap \Gamma$;
(U3) The partial derivatives of the coordinate functions φ_i and φ_i^{-1} are uniformly bounded by a constant K_0 (not depending on i).

For $1 < q(x) < \infty$, we define

$$\mathcal{B}_{n,k} = \sup_{x \in D^n} \frac{b_0(x)}{b_1(x)} r^{\frac{N-k}{q(x)} - \frac{N}{p(x)} + 1}(x). \tag{3.5}$$

Under these assumptions we can get the following trace theorem.

Theorem 3.1. *Let $p(x) \in C^{0,1}(\overline{\Omega})$, $1 < p^- \leqslant p(x) \ll N$, let $1 < q(x) < \infty$, such that $\frac{N-1}{q(x)} - \frac{N}{p(x)} + 1 \geqslant 0$ for all $x \in \Omega$, and r, v_0, v_1, w satisfy* (3.1)–(3.4).

If

$$W^{1,p(x)}(\Omega_n; v_0, v_1) \hookrightarrow L^{q(x)}(\Gamma_n; w) \quad \textit{is compact for every } n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \mathcal{B}_{n,1} = 0,$$

then the trace operator $W^{1,p(x)}(\Omega; v_0, v_1) \hookrightarrow L^{q(x)}(\partial\Omega; w)$ is compact.

If

$$W^{1,p(x)}(\Omega_n; v_0, v_1) \hookrightarrow L^{q(x)}(\Gamma_n; w) \quad \textit{is continuous for every } n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \mathcal{B}_{n,1} < \infty,$$

then the trace operator $W^{1,p(x)}(\Omega; v_0, v_1) \hookrightarrow L^{q(x)}(\partial\Omega; w)$ is continuous.

The proof of this theorem is given in Section 3.3. In order to prove it, we first give that there exists a linear bounded extension operator $\bar{\varepsilon} : W^{1,p(x)}(\Omega; v_0, v_1) \hookrightarrow W^{1,\bar{p}(x)}(\mathbb{R}^N; v_0, v_1)$, where $\Omega \subseteq \mathbb{R}^N$ with boundary satisfy (U1)–(U3) (Section 3.1), so we can transfer our trace theorem in $\mathbb{R}^N$ to general domain Ω, next, we give a special case form of Theorem 3.1 (Section 3.2).

3.1. Extension operator on $W^{1,p(x)}(\Omega; v_0, v_1)$

In this section, we follow the ideas of D.E. Edmunds and J. Rákosník's paper [10], which also can be found in L. Diening's paper [7], other extension operators can be seen in [6].

Lemma 3.1. *(See [10,12].) Let $1 < p(x) < \infty$, $-\infty < a_i < b_i < +\infty$, $i = 1, \dots, N-1$, $0 < b_N < +\infty$, $Q_+ = (a_1, b_1) \times \dots \times (a_{N-1}, b_{N-1}) \times (0, b_N)$, and let $u \in W^{1,p(x)}(Q_+)$. Define the extension Eu to $Q = (a_1, b_1) \times \dots \times (a_{N-1}, b_{N-1}) \times (-b_N, b_N)$ by*

$$Eu(x) = \begin{cases} u(x', x_N) & (x', x_N) \in Q_+, \\ u(x', -x_N) & (x', -x_N) \in Q_+. \end{cases}$$

Define Ep analogously, then $Eu \in W^{1,Ep(x)}(Q)$ and

$$|Eu|_{Ep,Q} \leqslant C|u|_{p,Q_+}, \qquad \big|\nabla(Eu)\big|_{Ep,Q} \leqslant C|\nabla u|_{p,Q_+}. \tag{3.6}$$

Definition 3.1. A mapping $T : \mathbb{R}^N \to \mathbb{R}^N$ is called bi-Lipschitz if there exists a constant L, and $1 \leqslant L < \infty$, such that

$$L^{-1}|x-y| \leqslant \big|T(x) - T(y)\big| \leqslant L|x-y|, \quad x, y \in \mathbb{R}^N.$$

To prove the extension theorem for Ω with boundary satisfy (U1)–(U3), we shall use the following property of bi-Lipschitz mapping.

Lemma 3.2. *(See [10].) Let Ω be a bounded domain in $\mathbb{R}^N$, $T : \mathbb{R}^N \to \mathbb{R}^N$ be a bi-Lipschitz mapping, $G = T^{-1}(\Omega)$ and $u \in W^{1,p(x)}(\Omega)$. Set $\bar{u} = u \circ T$, and set $\bar{p}$ analogously, then we have $\bar{u} \in W^{1,\bar{p}}(G)$ and*

$$\|\bar{u}\|_{1,\bar{p}(x),G} \leqslant C\|u\|_{1,p(x),\Omega},$$

where $C > 0$ depends only on N, $\operatorname{diam}(\Omega)$ and the Lipschitz constant L for T and T^{-1}.

Theorem 3.2. *Let Ω be a domain of $\mathbb{R}^N$ with boundary $\partial\Omega$ satisfying (U1)–(U3), $p \in L^\infty_+(\overline{\Omega})$, and the weight functions v_0, v_1 satisfy the condition (3.3), then there exist a function $\bar{p} \in L^\infty_+(\mathbb{R}^N)$ and a bounded linear extension operator $\bar{\varepsilon} : W^{1,p(x)}(\Omega; v_0, v_1) \hookrightarrow W^{1,\bar{p}(x)}(\mathbb{R}^N; v_0, v_1)$ such that $\bar{p}(x)|_\Omega = p(x)$, and*

$$\|\bar{\varepsilon}u\|_{1,\bar{p}(x),\mathbb{R}^N,v_0,v_1} \leqslant C\|u\|_{1,p(x),\Omega,v_0,v_1} \quad \text{for } u \in W^{1,p(x)}(\Omega; v_0, v_1),$$

where $C > 0$ is a constant independent of u. Moreover, there holds $\bar{p}^+ = p^+$, $\bar{p}^- = p^-$, and the extension $\bar{\varepsilon}u$ has support contained in $\{x \in \mathbb{R}^N\colon \operatorname{dist}(x, \Omega) \leqslant \beta\}$ for some positive number β.

Proof. From the assumptions of $\partial\Omega$, let $\{U_j\}_{j=1}^{+\infty}$ be the covering of the boundary $\partial\Omega$ which corresponds to the local description of $\partial\Omega$, more precisely, for each $j = 1, \dots, +\infty$, there is a local coordinate system (x', x_N) such that

$$U_j = \big\{(x', x_N)\colon\ |x_i| < \delta,\ i = 1, \dots, N-1,\ a_j(x') - \beta < x_N < a_j(x') + \beta\big\},$$
$$U_j \cap \Omega = \big\{x \in U_j\colon\ a_j(x') < x_N < a_j(x') + \beta\big\}$$

and

$$\big\{x \in \overline{U}_j\colon\ x_N < a_j(x')\big\} \cap \Omega = \emptyset,$$

where β, δ are some fixed positive numbers and $a_j \in C^{0,1}((-\delta, \delta)^{N-1})$ are the functions describing the boundary. Define the mappings

$$T_j : G = (-\delta, \delta)^{N-1} \times (-\beta, \beta) \to \mathbb{R}^N, \quad j = 1, \dots, +\infty,$$

by

$$T_j(x', x_N) = \big(x', x_N + a_j(x')\big),$$

then from (U1), we can assume that for all $x \in \partial\Omega$, there exists a neighborhood V_x of x in $\mathbb{R}^N$, such that the number of open set U_j intersecting V_x is finite and the multiplicity $\mathcal{M}$ of the covering $\{U_j\}_{j=1}^\infty$ is finite and $\mathcal{M} \leqslant \theta$; from (U2) and (U3), we know the T_j are bi-Lipschitz mappings and have uniformly bounded Lipschitz constant $L \geqslant K_0$.

Let $U_0 \subset \Omega$ be an open set such that $\overline{U}_0 \subset \Omega$ and $\overline{\Omega} \subset \bigcup_{j=0}^{+\infty} U_j$, let $\{\psi_j\}$ be a partition of unity subordinate to $\{U_j\}$, i.e. $\psi_j \in C_0^\infty(U_j)$, $0 \leqslant \psi_j \leqslant 1$, $\sum_{j=0}^{+\infty} \psi_j = 1$ on Ω and $\sup_{x \in U_j} |\nabla\psi_j| \leqslant C_0$, where C_0 is a constant only depending on N, δ, β (see V.I. Burenkov [4] or Remark 3.1).

Let $u \in W^{1,p(x)}(\Omega; v_0, v_1)$, we define the functions u_j by

$$u_j(x) = u(x)\psi_j(x), \quad x \in \Omega,\ j = 0, 1, 2, \dots, +\infty.$$

Since the weights v_0, v_1 satisfy (3.3), $r(x)$ is a bounded positive continuous function defined on $\mathbb{R}^N$, and $|U_j \cap \Omega| < \infty$, then there exists a positive constant C_1 such that $v_1(x) \leqslant C_1 v_0(x)$ in $U_j \cap \Omega$, and we have

$$\begin{aligned}\int_{U_j\cap\Omega} v_1(x)\left|\nabla u_j(x)\right|^{p(x)} dx &= \int_{U_j\cap\Omega} v_1(x)\left|\nabla u(x)\psi_j(x) + u(x)\nabla\psi_j(x)\right|^{p(x)} dx \\ &\leqslant \int_{U_j\cap\Omega} 2^{p^+}\left(v_1(x)\left|\nabla u(x)\psi_j(x)\right|^{p(x)} + v_1(x)\left|u(x)\nabla\psi_j(x)\right|^{p(x)}\right) dx \\ &\leqslant 2^{p^+}\int_{U_j\cap\Omega} v_1(x)\left|\nabla u(x)\right|^{p(x)} dx + 2^{p^+}(C_0)^{p^+}C_1\int_{U_j\cap\Omega} v_0(x)\left|u(x)\right|^{p(x)} dx \\ &< +\infty \end{aligned} \tag{3.7}$$

and

$$\int_{U_j\cap\Omega} v_0(x)\left|u_j(x)\right|^{p(x)} dx \leqslant \int_{U_j\cap\Omega} v_0(x)\left|u(x)\psi_j(x)\right|^{p(x)} dx \leqslant \int_{U_j\cap\Omega} v_0(x)\left|u(x)\right|^{p(x)} dx < +\infty,$$

so we can easily get $u_j \in W^{1,p(x)}(U_j \cap \Omega; v_0, v_1)$ and

$$\|u_j\|_{1,p(x),U_j\cap\Omega,v_0,v_1} \leqslant C_2\|u\|_{1,p(x),\Omega,v_0,v_1},$$

where C_2 depends on p, C_0 and C_1.

Set $G_+ = (-\delta,\delta)^{N-1} \times (0,\beta)$ and define the function g_j by

$$g_j(x) = \begin{cases} u_j(T_j(x)) & x \in G_+, \\ 0 & x \in \mathbb{R}^N_+ \setminus G_+, \end{cases}$$

where $j = 1, 2, \ldots, +\infty$, and $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N\colon\ x_N > 0\}$, similarly, set $p_j = p \circ T_j$, $v_{0j} = v_0 \circ T_j$ and $v_{1j} = v_j \circ T_j$ when $x \in G_+$, and 0 on $\mathbb{R}^N_+ \setminus G_+$, then we have for $\lambda > L^{1+N}|\nabla u_j(x)|_{p(x),U_j\cap\Omega,v_1}$

$$\int_{G_+} v_{1j}(x)\left|\frac{\nabla(g_j(x))}{\lambda}\right|^{p_j(x)} dx = \int_{G_+} v_1\bigl(T_j(x)\bigr)\left|\frac{\nabla(u_j(T_j(x)))}{\lambda}\right|^{p(T_j(x))} dx \leqslant \int_{U_j\cap\Omega} v_1(x)\left|\frac{\nabla u_j(x)}{|\nabla u_j|_{p(x),U_j\cap\Omega,v_1}}\right|^{p(x)} dx \leqslant 1, \tag{3.8}$$

similarly, let $\lambda > L^N|u_j|_{p(x),U_j\cap\Omega,v_0}$, we have

$$\int_{G_+} v_{0j}(x)\left|\frac{g_j(x)}{\lambda}\right|^{p(x)} dx \leqslant \int_{U_j\cap\Omega} v_0(x)\left|\frac{u_j(x)}{|u_j|_{p(x),U_j\cap\Omega,v_0}}\right|^{p(x)} dx \leqslant 1.$$

Let Eg_j be the extension of g_j as in Lemma 3.1, set Ev_{0j}, Ev_{1j} analogously, then

$$\int_{G\setminus G_+} Ev_{0j}(x)\left|Eg_j(x)\right|^{Ep_j(x)} dx = \int_{G_+} v_{0j}(x)\left|g_j(x)\right|^{p_j(x)} dx,$$

$$\int_{G\setminus G_+} Ev_{1j}(x)\left|\nabla\bigl(Eg_j(x)\bigr)\right|^{Ep_j(x)} dx = \int_{G_+} v_{1j}(x)\left|\nabla g_j(x)\right|^{p_j(x)} dx,$$

that is

$$\|Eg_j\|_{1,Ep_j(x),\mathbb{R}^N,Ev_{0j},Ev_{1j}} \leqslant 2\|g_j\|_{1,p_j(x),G_+,v_{0j},v_{1j}}, \tag{3.9}$$

also, it follows from the construction of E that $\operatorname{supp} Eg_j \subset G$.

We define the functions $\bar{p}_j$, $j = 1, \ldots, +\infty$ by

$$\bar{p}_j(x) = \begin{cases} p(x) & x \in \Omega, \\ Ep_j(T_j^{-1}(x)) & x \in U_j\setminus\Omega, \end{cases}$$

and extend $\bar{p}_j(x)$ to $\mathbb{R}^N$ preserving their upper and lower bounds, we define the functions $\bar{v}_{0j}$ and $\bar{v}_{1j}$, $j = 1, \ldots, +\infty$ by

$$\bar{v}_{0j}(x) = \begin{cases} Ev_{0j}(T_j^{-1}(x)) & x \in U_j\setminus\Omega, \\ v_0(x) & x \in \mathbb{R}^N\setminus(U_j\setminus\Omega); \end{cases}$$

$$\bar{v}_{1j}(x) = \begin{cases} Ev_{1j}(T_j^{-1}(x)) & x \in U_j\setminus\Omega, \\ v_1(x) & x \in \mathbb{R}^N\setminus(U_j\setminus\Omega). \end{cases}$$

Now, we define the functions $\bar{p}$, $\bar{v}_0$ and $\bar{v}_1$ by

$$\bar{p}(x) = \min_{1\leqslant j<+\infty}\{\bar{p}_j(x)\}, \quad x\in\mathbb{R}^N;$$

$$\bar{v}_0(x) = \min_{1\leqslant j<+\infty}\{\bar{v}_{0j}(x)\}, \quad x\in\mathbb{R}^N;$$

$$\bar{v}_1(x) = \min_{1\leqslant j<+\infty}\{\bar{v}_{1j}(x)\}, \quad x\in\mathbb{R}^N,$$

and the function $\bar{\varepsilon}u$ by

$$\bar{\varepsilon}u(x) = u_0(x) + \sum_{j=1}^{+\infty} Eg_j\big(T_j^{-1}(x)\big), \quad x\in\mathbb{R}^N,$$

where u_0 and $Eg_j\circ T_j^{-1}$ are extended by zero to the whole $\mathbb{R}^N$.

Clearly, $\bar{\varepsilon}u(x)|_\Omega = u(x)$. Since Eg_j and T_j^{-1} are bounded, the inequality $\frac{\max_{x\in U_j\backslash\Omega}\{v_1(x)\}}{\min_{x\in U_j\backslash\Omega}\{\bar{v}_1(x)\}} \leqslant C(\ell(U_j\backslash\Omega)) \leqslant C_\ell$ $(C_\ell>1)$ holds for all $j=1,2,\dots,+\infty$, and the estimates (3.7), (3.8) and (3.9), we obtain for $\lambda > (C_2\theta+1)^{\frac{1}{p^-}}L^{2+2N}\cdot |||u|||_{1,p(x),\Omega,v_0,v_1}$,

$$\begin{aligned}
\int_{\mathbb{R}^N} v_1(x)\left|\frac{\nabla(\bar{\varepsilon}u(x))}{(C_\ell)^{\frac{1}{\bar{p}^+}}\lambda}\right|^{\bar{p}(x)}dx &\leqslant \sum_{j=1}^{+\infty}\int_{U_j\backslash\Omega} v_1(x)\left|\frac{\nabla(\bar{\varepsilon}u(x))}{(C_\ell)^{\frac{1}{\bar{p}^+}}\lambda}\right|^{\bar{p}(x)}dx + \int_\Omega v_1(x)\left|\frac{\nabla(\bar{\varepsilon}u(x))}{\lambda}\right|^{\bar{p}(x)}dx\\
&= \sum_{j=1}^{+\infty}\int_{U_j\backslash\Omega} \frac{v_1(x)}{\bar{v}_1(x)}\bar{v}_1(x)\left|\frac{\nabla(\bar{\varepsilon}u(x))}{(C_\ell)^{\frac{1}{\bar{p}^+}}\lambda}\right|^{\bar{p}(x)}dx + \int_\Omega v_1(x)\left|\frac{\nabla(\bar{\varepsilon}u(x))}{\lambda}\right|^{\bar{p}(x)}dx\\
&\leqslant \sum_{j=1}^{+\infty}\int_{U_j\backslash\Omega} \bar{v}_1(x)\left|\frac{\nabla(\bar{\varepsilon}u(x))}{\lambda}\right|^{\bar{p}(x)}dx + \int_\Omega v_1(x)\left|\frac{\nabla(\bar{\varepsilon}u(x))}{\lambda}\right|^{\bar{p}(x)}dx\\
&\leqslant \sum_{j=1}^{+\infty}\int_{U_j\backslash\Omega} Ev_{1j}\big(T_j^{-1}(x)\big)\left|\frac{\nabla(Eg_j(T_j^{-1}(x)))}{\lambda}\right|^{Ep_j(T_j^{-1}(x))}dx + \int_\Omega v_1(x)\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)}dx\\
&\leqslant \sum_{j=1}^{+\infty}\int_{G\backslash G_+} Ev_{1j}(x)\left|\frac{\nabla(Eg_j(x))}{\lambda L^{-1-N}}\right|^{Ep_j(x)}dx + \int_\Omega v_1(x)\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)}dx\\
&= \sum_{j=1}^{+\infty}\int_{G_+} v_{1j}(x)\left|\frac{\nabla g_j(x)}{\lambda L^{-1-N}}\right|^{p_j(x)}dx + \int_\Omega v_1(x)\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)}dx\\
&= \sum_{j=1}^{+\infty}\int_{G_+} v_1\big(T_j(x)\big)\left|\frac{\nabla u_j(T_j(x))}{\lambda L^{-1-N}}\right|^{p(T_j(x))}dx + \int_\Omega v_1(x)\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)}dx\\
&\leqslant \sum_{j=1}^{+\infty}\int_{U_j\cap\Omega} v_1(x)\left|\frac{\nabla u_j(x)}{\lambda L^{-2-2N}}\right|^{p(x)}dx + \int_\Omega v_1(x)\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)}dx\\
&\leqslant \sum_{j=1}^{+\infty} C_2\left\{\int_{U_j\cap\Omega} v_1(x)\left|\frac{\nabla u(x)}{\lambda L^{-2-2N}}\right|^{p(x)}dx + \int_{U_j\cap\Omega} v_0(x)\left|\frac{u(x)}{\lambda L^{-2-2N}}\right|^{p(x)}dx\right\}\\
&\quad + \int_\Omega v_1(x)\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)}dx\\
&\leqslant (C_2\theta+1)\int_\Omega v_1(x)\left|\frac{\nabla u(x)}{\lambda L^{-2-2N}}\right|^{p(x)}dx + C_2\theta\int_\Omega v_0(x)\left|\frac{u(x)}{\lambda L^{-2-2N}}\right|^{p(x)}dx
\end{aligned}$$

$$\begin{aligned}&\leqslant (C_2\theta+1)\int_\Omega v_1(x)\left|\frac{\nabla u(x)}{(C_2\theta+1)^{\frac{1}{p^-}}\|\|u\|\|_{1,p(x),\Omega,v_0,v_1}}\right|^{p(x)}\\&\quad+v_0(x)\left|\frac{u(x)}{(C_2\theta+1)^{\frac{1}{p^-}}\|\|u\|\|_{1,p(x),\Omega,v_0,v_1}}\right|^{p(x)}dx\\&\leqslant 1,\end{aligned}\tag{3.10}$$

similarly, since $\frac{\max_{x\in U_j\backslash\Omega}\{v_0(x)\}}{\min_{x\in U_j\backslash\Omega}\{\bar v_0(x)\}}\leqslant C'(\ell(U_j\backslash\Omega))\leqslant C'_\ell$ ($C'_\ell>1$) for all $j=1,2,\dots,+\infty$, let $\lambda>\theta^{\frac{1}{p^-}}L^{2N}\|\|u\|\|_{1,p(x),\Omega,v_0,v_1}$, we have

$$\int_{\mathbb{R}^N} v_0(x)\left|\frac{\bar\varepsilon u(x)}{(C'_\ell)^{\frac{1}{\bar p^+}}\lambda}\right|^{\bar p(x)}dx\leqslant\theta\int_\Omega v_0(x)\left|\frac{u(x)}{\theta^{\frac{1}{p^-}}L^{2N}\|\|u\|\|_{1,p(x),\Omega,v_0,v_1}}\right|^{p(x)}dx\leqslant 1.\tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\|\|\bar\varepsilon u\|\|_{1,\bar p(x),\mathbb{R}^N,\bar v_0,\bar v_1}<C\|\|u\|\|_{1,p(x),\Omega,v_0,v_1}$$

where $C=\max\{(C'_\ell)^{\frac{1}{\bar p^+}}\theta^{\frac{1}{p^-}},(C_\ell)^{\frac{1}{\bar p^+}}(C_2\theta+1)^{\frac{1}{p^-}}L^2\}\cdot L^{2N}\leqslant(C_\ell)^{\frac{1}{\bar p^+}}(C'_\ell)^{\frac{1}{\bar p^+}}(C_2\theta+1)^{\frac{1}{p^-}}L^{(2N+1)}$ is a bounded constant independent of u. Since the two norms in the above inequality are equivalent, the proof is complete. □

Remark 3.1. We can construct the partition of unity subordinate to $\{U_j\}$ as follows: without lost of generality, we set $\delta=\beta$, and set $(U_j)_d=\{x\in U_j:\ \mathrm{dist}(x,\partial U_j)\geqslant d,\ d>0\}$, $j=1,2,\dots,+\infty$. From the assumptions of the boundary $\Gamma=\partial\Omega$, we can set $\{(U_j)_{\frac{\delta}{2}}\}$ also the covering of $\partial\Omega$, then we can choose U_0 such that $\Omega\subset\bigcup_{j=0}^{+\infty}\{(U_j)_{\frac{\delta}{2}}\}$ is also satisfied. Let $\eta_j\in C_0^\infty(U_j)$ be "cap-shaped" functions satisfying $0\leqslant\eta_j\leqslant 1$, $\eta_j=1$ on $(U_j)_{\frac{\delta}{2}}$, $\eta_j=0$ on $\mathbb{R}^N\backslash(U_j)_{\frac{\delta}{4}}$ and $\sup_{x\in(U_j)}|\nabla\eta_j(x)|\leqslant M\delta^{-1}$, where M is a constant depending only on N, then we have $1\leqslant\sum_{j=0}^\infty\eta_j\leqslant\mathcal{M}$ on $\bigcup_{j=0}^\infty(U_j)_{\frac{\delta}{2}}$. Moreover, let $\eta\in C^\infty(\mathbb{R}^N)$, $0\leqslant\eta\leqslant 1$, $\eta=1$ on Ω, $\eta=0$ on $\mathbb{R}^N\backslash(\bigcup_{j=0}^\infty(U_j)_{\frac{\delta}{2}})$, we can construct functions ψ_j by setting $\psi_j=\eta_j\eta(\sum_{j=0}^{+\infty}\eta_j)^{-1}$ on $\bigcup_{j=0}^\infty(U_j)_{\frac{\delta}{2}}$ and assuming that $\psi_j=0$ on $\mathbb{R}^N\backslash(\bigcup_{j=0}^{+\infty}(U_j)_{\frac{\delta}{2}})$, $j=0,1,2,\dots,+\infty$, since

$$|\nabla\psi_j|=\left|\frac{\nabla(\eta\eta_j)}{\sum_{j=0}^{+\infty}\eta_j}-\frac{\eta\eta_j\nabla(\sum_{j=0}^{+\infty}\eta_j)}{(\sum_{j=0}^\infty\eta_j)^2}\right|\leqslant\left|\frac{\nabla\eta\eta_j+\eta\nabla\eta_j}{\sum_{j=0}^{+\infty}\eta_j}\right|+\left|\frac{\eta\eta_j\sum_{j=0}^{+\infty}\nabla\eta_j}{(\sum_{j=0}^{+\infty}\eta_j)^2}\right|\leqslant 2M\delta^{-1}+\mathcal{M}M\delta^{-1}=C_0.$$

3.2. Compact traces on planes

The aim of this section is to prove a special case of Theorem 3.1, that is

Theorem 3.3. *Let $p(x)\in C^{0,1}(\overline{\Omega})$, $1<p^-\leqslant p(x)\ll N$, let $1<q(x)<\infty$, such that $\frac{N-k}{q(x)}-\frac{N}{p(x)}+1\geqslant 0$ for all $x\in\Omega$, and Γ be the intersection of the domain Ω with an $(N-k)$-dimensional plane. Denote $\Omega_n=D_n\cap\Omega$, $\Gamma_n=D_n\cap\Gamma$, $\Gamma^n=\Gamma\setminus\Gamma_n$ and r, v_0, v_1, w satisfy* (3.1)–(3.4).

If $W^{1,p(x)}(\Omega_n;v_0,v_1)\hookrightarrow L^{q(x)}(\Gamma_n;w)$ is compact for every $n\in\mathbb{N}$ and $\lim_{n\to\infty}\mathcal{B}_{n,k}=0$, then $W^{1,p(x)}(\Omega;v_0,v_1)\hookrightarrow L^{q(x)}(\Gamma;w)$ is compact.

If $W^{1,p(x)}(\Omega_n;v_0,v_1)\hookrightarrow L^{q(x)}(\Gamma_n;w)$ is continuous for every $n\in\mathbb{N}$ and $\lim_{n\to\infty}\mathcal{B}_{n,k}<\infty$, then $W^{1,p(x)}(\Omega;v_0,v_1)\hookrightarrow L^{q(x)}(\Gamma;w)$ is continuous.

Proof. We may assume that Γ has the form

$$\Gamma=\Omega\cap\mathbb{R}^{N-k}=\{x\in\Omega:\ x=(x',0),\ x'\in\mathbb{R}^{N-k}\}.$$

From the Section 3.1, we may assume $\Omega=\mathbb{R}^N$, i.e. $\Gamma=\mathbb{R}^{N-k}$, let $\Gamma_R^n=\{x\in\Gamma^n:\ |x|<R\}$, where $|x|$ denotes the maximum norm in $\mathbb{R}^{N-k}$, since $\Gamma_R^{\tilde n}$ is bounded, by the Besicovitch covering lemma, there exists a locally finite covering with cubes

$$Q^\gamma(x_j,r(x_j))=\{x\in\mathbb{R}^{N-k}:\ |x-x_j|<r(x_j)\},\quad j\in\mathbb{N},$$

such that

$$\sum_j\chi_{Q^\gamma(x_j,r(x_j))}(\eta)\leqslant\theta_0\quad\text{for every }\eta\in\mathbb{R}^{N-k},$$

where χ_{Q^γ} is the characteristic function on Q^γ and the constant $\theta_0 > 1$ only depends on the dimension $N-k$, we denote by $Q(x_j, r(x_j))$ the corresponding cube in $\mathbb{R}^N$ with same center and edge length, since $z \in Q(x_j, r(x_j))$ implies that the projection $\pi(z)$ on $\mathbb{R}^{N-k}$ is in $Q^\gamma(x_j, r(x_j))$, we get

$$\sum_j \chi_{Q(x_j,r(x_j))}(z) \leqslant \theta_0 \quad \text{for every } z \in \mathbb{R}^N,$$

with the same θ_0 as above, where χ_Q is the characteristic function on Q.

We define

$$I = \{ j \in \mathbb{N}:\ Q(x_j, r(x_j)) \cap D^{3\tilde{n}} \neq \emptyset \},$$

from (3.1) it follows that $Q(x_j, r(x_j)) \subset D^{\tilde{n}}$ if $j \in I$ and we get

$$\Gamma_R^n \subset \bigcup_{j\in I} Q(x_j, r(x_j)) \subset D^{\tilde{n}} \quad \text{for every } n \geqslant 3\tilde{n},$$

this shows that we can use the estimates (3.1)–(3.4) for $n \geqslant 3\tilde{n}$ and $j \in I$.

Now, because of $p(x) \in C^{0,1}(\Omega)$, $1 < p^- \leqslant p(x) \ll N$ and $\frac{N-k}{q(x)} - \frac{N}{p(x)} + 1 \geqslant 0$ for all $x \in \Omega$, from Proposition 2.8, we can let K_2 denote the Sobolev constant for trace operator in the cube $Q(0,1)$ center at the origin, i.e.

$$|u|_{q(x),Q^\gamma(0,1)} \leqslant K_2 \|u\|_{1,p(x),Q(0,1)}.$$

Then by translation to x and dilation by r, and by Proposition 2.5 we obtain

$$\begin{aligned}
|u|_{q(\eta),Q^\gamma(x,r(x))} &\leqslant C_1 r^{\frac{N-k}{q(x)}}(x)|u|_{q(y),Q^\gamma(0,1)} \\
&\leqslant C_1 K_2 r^{\frac{N-k}{q(x)}}(x)\|u\|_{1,p(y),Q(0,1)} \\
&\leqslant C_2 K_2 r^{\frac{N-k}{q(x)}}(x)\big(r^{-\frac{N}{p(x)}}(x)|u|_{p(z),Q(x,r(x))} + r^{-\frac{N}{p(x)}+1}(x)|\nabla u|_{p(z),Q(x,r(x))}\big) \\
&\leqslant C_2 K_2 r^{\frac{N-k}{q(x)} - \frac{N}{p(x)}+1}(x) \cdot \big(r^{-1}(x)|u|_{p(z),Q(x,r(x))} + |\nabla u|_{p(z),Q(x,r(x))}\big),
\end{aligned}$$

where $y \in Q(0,1)$, $\eta = x + r(x)y$ when $y \in Q^\gamma(0,1)$, and $z = x + r(x)y$ when $y \in Q(0,1)$ (actually, η is the project of z on $Q^\gamma(x, r(x))$).

From Propositions 2.6, 2.7 and (3.1)–(3.5), we get for $j \in I$ (here $\eta = x_j + r(x_j)y$ when $y \in Q^\gamma(0,1)$, and $z = x_j + r(x_j)y$ when $y \in Q(0,1)$)

$$\begin{aligned}
|u|_{q(\eta),Q^\gamma(x_j,r(x_j)),w(\eta)} &\leqslant b_0(x_j)|u|_{q(y),Q^\gamma(0,1)} \\
&\leqslant C_2 K_2 b_0(x_j) r^{\frac{N-k}{q(x_j)} - \frac{N}{p(x_j)}+1}(x_j)\big(r^{-1}(x_j)|u|_{p(z),Q(x_j,r(x_j))} + |\nabla u|_{p(z),Q(x_j,r(x_j))}\big) \\
&\leqslant C_2 K_2 \frac{b_0(x_j)}{b_1(x_j)} r^{\frac{N-k}{q(x_j)} - \frac{N}{p(x_j)}+1}(x_j)\bigg(\frac{b_1(x_j)}{r(x_j)}|u|_{p(z),Q(x_j,r(x_j))} + b_1(x_j)|\nabla u|_{p(z),Q(x_j,r(x_j))}\bigg) \\
&\leqslant C_2 K_2 \frac{b_0(x_j)}{b_1(x_j)} r^{\frac{N-k}{q(x_j)} - \frac{N}{p(x_j)}+1}(x_j)\big(C_r K_1 |u|_{p(z),Q(x_j,r(x_j)),v_0(z)} + |\nabla u|_{p(z),Q(x_j,r(x_j)),v_1(z)}\big) \\
&\leqslant C_2 K_2 \max\{C_r K_1; 1\} \mathcal{B}_{n,k} \|u\|_{1,p(z),Q(x_j,r(x_j)),v_0(z),v_1(z)} \\
&\leqslant C \mathcal{B}_{n,k} |||u|||_{1,p(z),Q(x_j,r(x_j)),v_0(z),v_1(z)} \\
&= C \frac{1}{(C_{x_j})^{\frac{1}{p^+}}} \mathcal{B}_{n,k} |||u|||_{1,p(z),\mathbb{R}^N,v_0(z),v_1(z)}
\end{aligned}$$

where $C_{x_j} = \big(\frac{|||u|||_{1,p(z),\mathbb{R}^N,v_0(z),v_1(z)}}{|||u|||_{1,p(z),Q(x_j,r(x_j)),v_0(z),v_1(z)}}\big)^{p^+} > 1$, and $C > 0$ is a bounded constant independent of u. From the above estimate, let $\lambda_1 = |||u(z)|||_{1,p(z),Q(x_j,r(x_j)),v_0(z),v_1(z)}$ and $\lambda_2 = |||u(z)|||_{1,p(z),\mathbb{R}^N,v_0(z),v_1(z)}$, we have for $\lambda \geqslant C\mathcal{B}_{n,k}\lambda_1$ and $j \in I$

$$\int_{Q^\gamma(x_j,r(x_j))} w(\eta)\left|\frac{u(\eta)}{\lambda}\right|^{q(\eta)} d\eta \leqslant 1 = \int_{Q(x_j,r(x_j))} v_0(z)\left|\frac{u(z)}{\lambda_1}\right|^{p(z)} + v_1(z)\left|\frac{\nabla u(z)}{\lambda_1}\right|^{p(z)} dz,$$

so we can get the inequality

$$\int_{Q^\gamma(x_j,r(x_j))} w(\eta)\left|\frac{u(\eta)}{(C_{x_j})^{\frac{1}{q^-}}\lambda}\right|^{q(\eta)} d\eta \leqslant \int_{Q(x_j,r(x_j))} v_0(z)\left|\frac{u(z)}{(C_{x_j})^{\frac{1}{p^+}}\lambda_1}\right|^{p(z)} + v_1(z)\left|\frac{\nabla u(z)}{(C_{x_j})^{\frac{1}{p^+}}\lambda_1}\right|^{p(z)} dz,$$

because we have for all $j \in I$

$$\int\limits_{Q^{\gamma}(x_j,r(x_j))} w(\eta)\left|\frac{u(\eta)}{(C_{x_j})^{\frac{1}{q^-}}\lambda}\right|^{q(\eta)} d\eta \leqslant \frac{1}{C_{x_j}} \int\limits_{Q^{\gamma}(x_j,r(x_j))} w(\eta)\left|\frac{u(\eta)}{\lambda}\right|^{q(\eta)} d\eta$$

and

$$\int\limits_{Q(x_j,r(x_j))} v_0(z)\left|\frac{u(z)}{(C_{x_j})^{\frac{1}{p^+}}\lambda_1}\right|^{p(z)} + v_1(z)\left|\frac{\nabla u(z)}{(C_{x_j})^{\frac{1}{p^+}}\lambda_1}\right|^{p(z)} dz \geqslant \frac{1}{C_{x_j}} \int\limits_{Q(x_j,r(x_j))} v_0(z)\left|\frac{u(z)}{\lambda_1}\right|^{p(z)} + v_1(z)\left|\frac{\nabla u(z)}{\lambda_1}\right|^{p(z)} dz.$$

Set $C_x = \max\{C_{x_j},\, j \in I\}$, taking the sum over $j \in I$ in the above estimate, we obtain for $n \geqslant 3\tilde{n}$,

$$\begin{aligned}
\int\limits_{\Gamma_R^n} w(\eta)\left|\frac{u(\eta)}{(\theta_0)^{\frac{p^+}{p^-}} C_x\lambda}\right|^{q(\eta)} d\eta &\leqslant \sum_{j\in I} \int\limits_{Q^{\gamma}(x_j,r(x_j))} w(\eta)\left|\frac{u(\eta)}{(\theta_0)^{\frac{p^+}{p^-}} C_{x_j}\lambda}\right|^{q(\eta)} d\eta \\
&\leqslant \sum_{j\in I} \int\limits_{Q^{\gamma}(x_j,r(x_j))} w(\eta)\left|\frac{u(\eta)}{[(\theta_0)^{\frac{p^+}{p^-}} C_{x_j}]^{\frac{1}{q^-}}\lambda}\right|^{q(\eta)} d\eta \\
&\leqslant \sum_{j\in I} \int\limits_{Q(x_j,r(x_j))} v_0(z)\left|\frac{u(z)}{[(\theta_0)^{\frac{p^+}{p^-}} C_{x_j}]^{\frac{1}{p^+}}\lambda_1}\right|^{p(z)} + v_1(z)\left|\frac{\nabla u(z)}{[(\theta_0)^{\frac{p^+}{p^-}} C_{x_j}]^{\frac{1}{p^+}}\lambda_1}\right|^{p(z)} dz \\
&= \sum_{j\in I} \int\limits_{Q(x_j,r(x_j))} v_0(z)\left|\frac{u(z)}{(\theta_0)^{\frac{1}{p^-}}\lambda_2}\right|^{p(z)} + v_1(z)\left|\frac{\nabla u(z)}{(\theta_0)^{\frac{1}{p^-}}\lambda_2}\right|^{p(z)} dz \\
&\leqslant \theta_0 \int\limits_{\mathbb{R}^N} v_0(z)\left|\frac{u(z)}{(\theta_0)^{\frac{1}{p^-}}\lambda_2}\right|^{p(z)} + v_1(z)\left|\frac{\nabla u(z)}{(\theta_0)^{\frac{1}{p^-}}\lambda_2}\right|^{p(z)} dz \\
&\leqslant \int\limits_{\mathbb{R}^N} v_0(z)\left|\frac{u(z)}{\lambda_2}\right|^{p(z)} + v_1(z)\left|\frac{\nabla u(z)}{\lambda_2}\right|^{p(z)} dz \leqslant 1,
\end{aligned}$$

that is

$$|u|_{q(\eta),\Gamma_R^n,w(\eta)} \leqslant (\theta_0)^{\frac{p^+}{p^-}} C_x \cdot C\mathcal{B}_{n,k} |||u|||_{1,p(z),\mathbb{R}^N,v_0(z),v_1(z)}.$$

Since the right-hand side is independent of R, we can take the limit $R \to \infty$, and get

$$|u|_{q(\eta),\Gamma^n,w(\eta)} \leqslant (\theta_0)^{\frac{p^+}{p^-}} C_x \cdot C\mathcal{B}_{n,k} |||u|||_{1,p(z),\mathbb{R}^N,v_0(z),v_1(z)},$$

and θ_0, C_x, C are positive constants, we apply Theorem 2.1, if $\lim_{n\to\infty} \mathcal{B}_{n,k} = 0$, we see that condition (2.3) is fulfilled and the compactness of trace operator $W^{1,p(x)}(\mathbb{R}^N; v_0, v_1) \hookrightarrow L^{q(x)}(\Gamma; w)$ follows; if $\lim_{n\to\infty} \mathcal{B}_{n,k} < \infty$, then (2.5) holds and the trace operator $W^{1,p(x)}(\mathbb{R}^N; v_0, v_1) \hookrightarrow L^{q(x)}(\Gamma; w)$ is continuous. □

If we take Γ in Theorem 3.3 be the intersection of Ω with $\mathbb{R}^N$, we can obtain the following corollary.

Corollary 3.1. *Let $p(x) \in C^{0,1}(\overline{\Omega})$, $1 < p^- \leqslant p(x) \ll N$, let $1 < q(x) < \infty$, such that $\frac{N}{q(x)} - \frac{N}{p(x)} + 1 \geqslant 0$ for all $x \in \Omega$, and r, v_0, v_1, w satisfy* (3.1)–(3.4).

If $W^{1,p(x)}(\Omega_n; v_0, v_1) \hookrightarrow L^{q(x)}(\Omega_n; w)$ is compact for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathcal{B}_{n,0} = 0$, then $W^{1,p(x)}(\Omega; v_0, v_1) \hookrightarrow L^{q(x)}(\Omega; w)$ is compact.

If $W^{1,p(x)}(\Omega_n; v_0, v_1) \hookrightarrow L^{q(x)}(\Omega_n; w)$ is continuous for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathcal{B}_{n,0} < \infty$, then $W^{1,p(x)}(\Omega; v_0, v_1) \hookrightarrow L^{q(x)}(\Omega; w)$ is continuous.

3.3. Proof of Theorem 3.1

In this section, we use the same method of the Section 3.2.

We first consider the case $\Omega = \mathbb{R}^N$, since $\Gamma = \partial\Omega$ is a $(N-1)$-dimensional submanifold in $\mathbb{R}^N$ satisfying (U1)–(U3), denote $B_i^{\gamma} = B_i \cap \mathbb{R}^{N-1} \times \{0\}$, and let $\tilde{r} = r \circ \varphi_i^{-1}$ be the weight on B_i induced by r, then there exists a locally finite covering of B_i^{γ} with cubes $Q^{\gamma}(\xi_j, \tilde{r}(\xi_j))$ in $\mathbb{R}^{N-1}$, $\xi_i \in B_i^{\gamma}$. As in Section 3.2, we denote by $Q(\xi_j, \tilde{r}(\xi_j))$ the corresponding cubes in $\mathbb{R}^N$ and we have

$$\sum_j \chi_{Q(\xi_j,\tilde{r}(\xi_j))}(\zeta) \leqslant \theta_1, \quad \text{for every } \zeta \in \mathbb{R}^N,$$

where $\theta_1 > 1$ only depends on N. Next we write $\tilde{v}_0$, $\tilde{v}_1$, $\tilde{w}$, $\tilde{b}_0$, $\tilde{b}_1$ $\tilde{p}$, $\tilde{q}$, $\tilde{u}$ for the transformed functions on B_i, i.e. $\tilde{v}_0 = v_0 \circ \varphi_i^{-1}$ and so on.

From the assumption (U3), we obtain the following mapping properties for φ_i.

Lemma 3.3. *(See K. Pflüger's papers [22,24].) If* (U3) *holds, then*

(i) *The Jacobi determinants of φ_i and φ_i^{-1} are uniformly bounded $K_3 = K_0^N N!$;*
(ii) *For every $x \in U_i \cap \Gamma^{\tilde{n}}$, we have*

$$\varphi_i^{-1}\big[Q\big(\xi, \tilde{r}(\xi)\big)\big] \subset Q\big(x, K_4 r(x)\big), \quad \text{where } \xi = \varphi_i(x),\ K_4 = NK_0.$$

In the sequel, we can assume without restriction that the constant $K_4 = 1$, and these propositions guarantee that the following estimates analogous to (3.2)–(3.4) are valid

$$C_r^{-1} \leqslant \tilde{r}(\xi)/\tilde{r}(\zeta) \leqslant C_r, \qquad \text{for every } \xi = \varphi_i(x),\ x \in U_i \cap \Gamma^{\tilde{n}},\ \zeta \in Q\big(\xi, \tilde{r}(\xi)\big), \tag{3.12}$$

$$\big|\tilde{v}_1(\zeta)\big|^{\frac{1}{\tilde{p}(\zeta)}} \tilde{r}^{-1}(\zeta) \leqslant K_1 \big|\tilde{v}_0(\zeta)\big|^{\frac{1}{\tilde{p}(\zeta)}}, \qquad \text{for a.e. } \zeta = \varphi_i(x),\ x \in U_i \cap D^{\tilde{n}}, \tag{3.13}$$

$$\big|\tilde{w}(\zeta)\big|^{\frac{1}{\tilde{p}(\zeta)}} \leqslant \tilde{b}_0(\xi), \tilde{b}_1(\xi) \leqslant \big|\tilde{v}_1(\zeta)\big|^{\frac{1}{\tilde{p}(\zeta)}}, \qquad \text{for every } \xi = \varphi_i(x),\ x \in U_i \cap \Gamma^{\tilde{n}},\ \text{a.e. } \zeta \in Q\big(\xi, \tilde{r}(\xi)\big), \tag{3.14}$$

where C_r and K_1 are from (3.2) and (3.3).

As in Section 3.2, we first prove the following estimate ($n \geqslant 3\tilde{n}$)

$$|u|_{q(\eta), U_i \cap \Gamma_R^n, w(\eta)} \leqslant C\mathcal{B}_{n,1} \|u\|_{1,p(z),U_i,v_0(z),v_1(z)}. \tag{3.15}$$

By Proposition 2.6 and (3.12)–(3.14),

$$\begin{aligned}
|u|_{q(\eta), U_i \cap \Gamma_R^n, w(\eta)} &\leqslant K_3 \sum_j |\tilde{u}|_{\tilde{q}(\xi), Q^\gamma(\xi_j, \tilde{r}(\xi_j)), \tilde{w}(\xi)} \\
&\leqslant K_3 C_2 K_2 \sum_j \left(\frac{\tilde{b}_0(\xi_j)}{\tilde{b}_1(\xi_j)} \tilde{r}^{\frac{N-1}{\tilde{q}(\xi_j)} - \frac{N}{\tilde{p}(\xi_j)} + 1}(\xi_j) \right) \big(C_r K_1 |\tilde{u}|_{\tilde{p}(\zeta), Q(\xi_i, \tilde{r}(\xi_j)), \tilde{v}_0(\zeta)} + |\nabla \tilde{u}|_{\tilde{p}(\zeta), Q(\xi_j, \tilde{r}(\xi_j)), \tilde{v}_1(\zeta)} \big),
\end{aligned}$$

where $\eta \in U_i \cap \Gamma_R^n$, $\xi = \varphi_i(\eta) \in Q^\gamma(\xi_j, \tilde{r}(\xi_j))$ and $\zeta \in Q(\xi_j, \tilde{r}(\xi_j))$. Since $\tilde{b}_0(\xi_j) = b_0(x_j)$, $\tilde{b}_1(\xi_j) = b_1(x_j)$, $\tilde{r}(\xi_j) = r(x_j)$, $\tilde{p}(\xi_j) = p(x_j)$, $\tilde{q}(\xi_j) = q(x_j)$ with $x_j \in U_i \cap \Gamma_R^n \subset D_R^n$, we finally obtain (setting $z = \varphi_i^{-1}(\zeta)$)

$$\begin{aligned}
|u|_{q(\eta), U_i \cap \Gamma_R^n, w(\eta)} &\leqslant K_3 C_2 K_2 \theta_1 \mathcal{B}_{n,1} \max\{C_r K_1, 1\} \cdot \big(|\tilde{u}|_{\tilde{p}(\zeta), B_i, \tilde{v}_0(\zeta)} + |\nabla \tilde{u}|_{\tilde{p}(\zeta), B_i, \tilde{v}_1(\zeta)} \big) \\
&\leqslant C\mathcal{B}_{n,1} \big(|u|_{p(z), U_i, v_0(z)} + |\nabla u|_{p(z), U_i, v_1(z)} \big) \\
&\leqslant C\mathcal{B}_{n,1} \|u\|_{1,p(z),U_i,v_0(z),v_1(z)},
\end{aligned}$$

where C depends on K_0, K_1, K_2, K_3, C_r and θ_1, so (3.15) is proved.

Now, let $I = \{i \in \mathbb{N}\colon\ U_i \cap \Gamma^{\tilde{n}} \neq \emptyset\}$, then from (U1) and (3.15), similar to the proof of Theorem 3.3, we can obtain

$$|u|_{q(\eta), \Gamma_R^n, w(\eta)} \leqslant (\theta_1)^{\frac{p^+}{p^-}} C' \mathcal{B}_{n,1} |||u|||_{1,p(z),\mathbb{R}^N,v_0(z),v_1(z)},$$

where C' is a bounded constant independent of u, in the limit $R \to \infty$, we obtain for $n \geqslant 3\tilde{n}$

$$\big|u(\eta)\big|_{q(\eta), \Gamma^n, w(\eta)} \leqslant (\theta_1)^{\frac{p^+}{p^-}} C' \mathcal{B}_{n,1} \big\|u(z)\big\|_{1,p(z),\mathbb{R}^N,v_0(z),v_1(z)}.$$

Since θ_1, C' are positive constants, we can apply Theorem 2.1 again to the trace operator $W^{1,p(x)}(\mathbb{R}^N; v_0, v_1) \hookrightarrow L^{q(x)}(\Gamma; w)$, and for Section 3.1, by extension, we can get the results for general domains $\Omega \subseteq \mathbb{R}^N$.

Remark 3.2. (1) Conditions (U1)–(U3) are necessary to control the behaviour of $\partial\Omega$ at infinity, and from K. Pflüger's papers [22,24], we know that these conditions are satisfied for any compact Lipschitz-submanifold of $\mathbb{R}^N$, for unbounded domain, the infinite cylinder $Q \times \mathbb{R}$, where $Q \subseteq \mathbb{R}^N$ is smooth and bounded, obviously satisfies (U1)–(U3).

(2) If weights v_0, v_1, w are positive continuous functions defined in $\mathbb{R}^N$, then we can find a constant $C_n \geqslant 1$ such that for every $x \in \overline{\Omega}_n$,

$$C_n^{-1} \leqslant v_0(x), v_1(x), w(x) \leqslant C_n,$$

then, the condition $W^{1,p(x)}(\Omega_n; v_0, v_1) \hookrightarrow L^{q(x)}(\Gamma_n; w)$ is compact for every $n \in \mathbb{N}$ can be replaced by the condition $W^{1,p(x)}(\Omega_n) \hookrightarrow L^{q(x)}(\Gamma_n)$ is compact for every $n \in \mathbb{N}$.

(3) When Ω is a bounded domain, the operator $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Gamma)$ has been studied by X.L. Fan's papers [13,18].

4. Example

Example. Let $\Omega \subseteq \mathbb{R}^N$ be an unbounded domain with noncompact boundary satisfying (U1)–(U3), $p(x) \in C^{0,1}(\overline{\Omega})$ and $1 < p^- \leqslant p(x) \ll N$, assume that there are real functions $\alpha(x)$, $\beta(x)$ defined in $\mathbb{R}^N$, and $-N < \alpha(x), \beta(x), \beta(x) - p(x) < N(p(x) - 1)$ in Ω. Define $w(x) = (1 + |x|)^{\alpha(x)}$, $v_0(x) = (1 + |x|)^{\beta(x) - p(x)}$ and $v_1(x) = (1 + |x|)^{\beta(x)}$, then:

(1) If $1 < q(x) < \infty$, $\frac{\alpha(x)}{q(x)} - \frac{\beta(x)}{p(x)} + \frac{N-1}{q(x)} - \frac{N}{p(x)} + 1 \leqslant 0$ and $\frac{N-1}{q(x)} - \frac{N}{p(x)} + 1 \geqslant 0$ for all $x \in \Omega$, the trace operator $W^{1,p(x)}(\Omega; v_0, v_1) \hookrightarrow L^{q(x)}(\partial\Omega; w)$ is continuous; and if the two inequalities above are replaced by $\operatorname{ess\,sup}_{x \in \Omega}(\frac{\alpha(x)}{q(x)} - \frac{\beta(x)}{p(x)} + \frac{N-1}{q(x)} - \frac{N}{p(x)} + 1) < 0$ and $\operatorname{ess\,inf}_{x \in \Omega}(\frac{N-1}{q(x)} - \frac{N}{p(x)} + 1) > 0$, the corresponding trace operator is compact.

(2) If $1 < q(x) < \infty$, $\frac{\alpha(x)}{q(x)} - \frac{\beta(x)}{p(x)} + \frac{N}{q(x)} - \frac{N}{p(x)} + 1 \leqslant 0$ and $\frac{N}{q(x)} - \frac{N}{p(x)} + 1 \geqslant 0$ for all $x \in \Omega$, the trace operator $W^{1,p(x)}(\Omega; v_0, v_1) \hookrightarrow L^{q(x)}(\Omega; w)$ is continuous; and if the two inequalities above are replaced by $\operatorname{ess\,sup}_{x \in \Omega}(\frac{\alpha(x)}{q(x)} - \frac{\beta(x)}{p(x)} + \frac{N}{q(x)} - \frac{N}{p(x)} + 1) < 0$ and $\operatorname{ess\,inf}_{x \in \Omega}(\frac{N}{q(x)} - \frac{N}{p(x)} + 1) > 0$, the corresponding trace operator is compact.

Proof. Define D_n, D^n and Ω_n as above, we can choose

$$r(x) = \begin{cases} \frac{2}{9}(1 + |x|) & |x| < 2, \\ \frac{2}{3} & |x| \geqslant 2, \end{cases} \qquad b_0(x) = \frac{4}{3}\big(1 + |x|\big)^{\frac{\alpha(x)}{q(x)}}, \qquad b_1(x) = \frac{2}{3}\big(1 + |x|\big)^{\frac{\beta(x)}{p(x)}},$$

then we have $r(x) \leqslant (|x| + 1)/3$ for every $x \in D^{\tilde{n}}$, $\tilde{n} \geqslant 2$, and (3.2)–(3.4) are satisfied.

For all $x \in \Omega$, if $\frac{\alpha(x)}{q(x)} - \frac{\beta(x)}{p(x)} + \frac{N-1}{q(x)} - \frac{N}{p(x)} + 1 \leqslant 0$ then $\lim_{n \to \infty} \mathcal{B}_{n,1} < \infty$, and if $\frac{N-1}{q(x)} - \frac{N}{p(x)} + 1 \geqslant 0$, from Remark 3.2(2), $W^{1,p(x)}(\Omega_n; v_0, v_1) \hookrightarrow L^{q(x)}(\Gamma_n; w)$ is continuous for every $n \in \mathbb{N}$, and from Theorem 3.1, $W^{1,p(x)}(\Omega; v_0, v_1) \hookrightarrow L^{q(x)}(\partial\Omega; w)$ is continuous. If these two inequalities are replaced by $\operatorname{ess\,sup}_{x \in \Omega}(\frac{\alpha(x)}{q(x)} - \frac{\beta(x)}{p(x)} + \frac{N-1}{q(x)} - \frac{N}{p(x)} + 1) < 0$ and $\operatorname{ess\,inf}_{x \in \Omega}(\frac{N-1}{q(x)} - \frac{N}{p(x)} + 1) > 0$, then we can get $\lim_{n \to \infty} \mathcal{B}_{n,1} = 0$ and $W^{1,p(x)}(\Omega_n; v_0, v_1) \hookrightarrow L^{q(x)}(\Gamma_n; w)$ is compact for every $n \in \mathbb{N}$, from Theorem 3.1, we conclude that $W^{1,p(x)}(\Omega; v_0, v_1) \hookrightarrow L^{q(x)}(\partial\Omega; w)$ is compact.

Applying Corollary 3.1 we can proved part (2) in a similar way. □

Acknowledgment

The author would like to thank Professor Xianling Fan for kind help and clear suggestions, and reviewer's useful comments.

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